Question: On which interval do you think this series converges? *(Why?)*

$$\sum_{k=1}^{\infty} \frac{x^k}{k}, \quad a_k = \frac{x^k}{k} \text{ for } k \geq 1$$

apply the ratio test:

$$L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{|x|^{k+1}}{(k+1)} \cdot \frac{k}{|x|^k}$$

$$= |x| \cdot \lim_{k \to \infty} \frac{k}{k+1} = |x|$$

set $L<1 \longleftrightarrow |x|<1$ to get convergence

$\longleftrightarrow -1<x<1$

Radius and I.C.

To find the radius of convergence of a power series in **standard form**, use the ratio or root test to find:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ or } L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

(i) If $L=0$, then $R=\infty$ and I.C. is all real numbers.

(ii) If $L=\infty$, then $R=0$ and I.C. is just $x=c$.

(iii) If L is positive and finite, then $R=1/L$, and the series converges for $|x-c|<R$. You must also check the endpoints.

Cautionary note!

If the power series is not in standard form, you may find the radius using one of these two methods:

1. Rewrite the series in standard form, and then evaluate the limit.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ or } L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

2. Put the *entire series* (with the “x” terms as well!) into the ratio or root test, then solve to find where the resulting limit is less than 1.

Example 1.1:

Find the radius and interval of convergence for the power series.

$$S = \sum_{k=1}^{\infty} \frac{(x+1)^k}{6^k}$$, let $a_k = \frac{(x+1)^k}{6^k}$ for $k \geq 1$

→ use the ratio test:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{|x+1|^{k+1}}{6^{k+1}} \cdot \frac{6^k}{|x+1|^k}$$

$$= \lim_{k\to\infty} \frac{|x+1|}{6} = \frac{|x+1|}{6}$$

→ we want that $L < 1$ to get convergence:

So $|x+1| < 6$

→ to find the interval of convergence:

$$-6 < x+1 < 6$$

$$\Leftrightarrow -7 < x < 5$$

→ we need to check the endpoints:

Plug in $x=-7$: $\sum_{k=1}^{\infty} \frac{(-6)^k}{6^k} = \sum_{k=1}^{\infty} (-1)^k$, diverges

Plug in $x=5$: $\sum_{k=1}^{\infty} \frac{6^k}{6^k} = \sum_{k=1}^{\infty} 1$, diverges as well

$\rightarrow$ So the interval of convergence is $(-7, 5)$, the radius is 6

An alternate approach to the problem (appended after the lecture):

→ write

$$S = \sum_{k=1}^{\infty} a_k (x-c)^k,$$ for $a_k = \frac{1}{6^k}$ and $c = -1$

so that the series is in a "standard form"

→ To find the radius of convergence (R):

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{6^k}{6^{k+1}} = \frac{1}{6},$$

so that $R = 1/L = 6$

→ Then the interval of convergence is found by

$$|x-c| = |x+1| < R = 6$$

$$\longleftrightarrow \quad -6 < x+1 < 6$$

Now proceed as we originally did in the lecture, checking at the endpoints, to see that the I.C. is $(-7,5)$

To answer the question asked during class about an intuitive meaning of the radius of convergence (R) of a power series:

→ it is $\frac{1}{2}$ the length of the largest interval for x on which the power series converges

Source: https://en.wikipedia.org/wiki/Radius_of_convergence

→ for example, in Ex 1.1 above, the IC was $(-7,5)$ and $R=6$ so that
$R = \frac{1}{2}(5-(-7))$, where the series is convergent for all $x \in (-7,5)$.

Example 1.2:

Find the radius and interval of convergence for the power series.

$$\sum_{k=1}^{\infty} \frac{(4-3x)^k}{\sqrt{2k+5}} = \sum_{k=1}^{\infty} \frac{(-3)^k \left(x-\frac{4}{3}\right)^k}{\sqrt{2k+5}}, \text{ take } a_k = \frac{(-3)^k}{\sqrt{2k+5}}$$

and take $c = 4/3$

$$= \sum_{k=1}^{\infty} a_k (x-c)^k$$

$\rightarrow$ look to find the radius of convergence:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{3^{k+1}}{\sqrt{2k+7}} \cdot \frac{\sqrt{2k+5}}{3^k}$$

$\sqrt{2(k+1)+5}$

$$= \lim_{k \to \infty} \frac{3 \cdot \sqrt{2k+5}}{\sqrt{2k+7}} = 3$$

So the radius is $R = \frac{1}{L} = \frac{1}{3}$

$\rightarrow$ we can solve for the interval of conv as

$$|x - 4/3| < R = \frac{1}{3}$$

$$-\frac{1}{3} < x - \frac{4}{3} < \frac{1}{3}$$

$$\Longleftrightarrow \quad 1 < x < 5/3$$

Plug in $x=1$: $\sum_{k=1}^{\infty} \frac{(-3)^k(-\frac{1}{3})^k}{\sqrt{2k+5}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+5}} = S_1$

$\rightarrow$ diverges by the LCT, compare with $b_k = \frac{1}{\sqrt{k}}$

$$\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{2k+5}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} > 0$$

So both S_1 and $\sum_{k=1}^{\infty} b_k$ diverge.

Plug in $x=5/3$: $S_2 = \sum_{k=1}^{\infty} \frac{(-3)^k(\frac{1}{3})^k}{\sqrt{2k+5}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{2k+5}}$

→ apply the AST:

- check that the sequence is decreasing:

$$\frac{1}{\sqrt{2k+5}} < \frac{1}{\sqrt{2(k+1)+5}} \text{ for all } k \geq 1 \checkmark$$

- check that

$$\lim_{k\to\infty} \frac{1}{\sqrt{2k+5}} = 0 \checkmark$$

- so S_2 converges (conditionally)

→ interval of convergence: $(1, 5/3]$

Example:

Find the radius and interval of convergence for

$$\sum_{k=\cancel{0}\,1}^{\infty} \frac{2^k}{k^3}(x-1)^k$$

, here $c=1$ and $a_k = \frac{2^k}{k^3}$ for $k \geq 1$

→ find the radius using the ratio test:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)^3} \cdot \frac{k^3}{2^k} = 2$$

A. R=1/2, I.C.=[1/2,3/2]

~~B.~~ R=2, I.C.=(-1,3)

~~C.~~ R=1/2, I.C.=[1/2,3/2)

~~D.~~ R=2, I.C.=[-1,3]

$R = \frac{1}{L} = \frac{1}{2}$

$\rightarrow$ interval of convergence:

$$|x-1| < \frac{1}{2} \longleftrightarrow -\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\longleftrightarrow \frac{1}{2} < x < 3/2$$

Plug in $x = \frac{1}{2}$:

$$S_1 = \sum_{k=1}^{\infty} \frac{2^k \left(-\frac{1}{2}\right)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$$

we get absolute convergence (p-series with $p=3$)

Plug in $x = 3/2$:

$$S_2 = \sum_{k=1}^{\infty} \frac{2^k \left(\frac{1}{2}\right)^k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^3},$$ converges as a p-series with $p = 3$

$\rightarrow$ so the interval of convergence is

$$\left[\frac{1}{2}, \frac{3}{2}\right]$$

Differentiation and Integration

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} \cdot x^k$$

$$\int\left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

The radius and interval of convergence are preserved under termwise differentiation and integration.

Example 2.1:

Find a power series expansion for the function

$$f(x) = \ln(x+1)$$

For what values of x is this formula valid?

$|x| < 1$

$$\ln(x+1) = \int_0^x \frac{dt}{t+1},$$

and we can expand $(t+1)^{-1}$ by a geometric series $|t| < 1$

$$= \int_0^x \sum_{N=0}^{\infty} (-1)^N \cdot t^N \, dt$$

$$= \sum_{N=0}^{\infty} (-1)^N \int_0^x t^N \, dt$$

$$= \sum_{N=0}^{\infty} (-1)^N \left. \frac{t^{N+1}}{N+1} \right|_0^x = \sum_{N=0}^{\infty} \frac{(-1)^N x^{N+1}}{N+1}$$

$$M = N+1$$

$$= \sum_{N=1}^{\infty} (-1)^{N-1} \frac{x^N}{N}$$

$$\sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$$

Example 2.2:

Find a power series expansion for the function

$$g(x) = \frac{1}{(1-x)^2}$$

For what values of x is this formula valid? *(Explain briefly.)*

Notice that $g(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right]$, and we

can expand $\frac{1}{1-x}$ by the geometric series

formula when $|x| < 1$:

$r = x$

$$g(x) = \frac{d}{dx}\left[\sum_{N=0}^{\infty} x^N\right] = \sum_{N=0}^{\infty} \frac{d}{dx}\left[x^N\right]$$

$$= \sum_{N=0}^{\infty} \underbrace{N}_{=0 \text{ at } N=0} \cdot x^{N-1} \qquad = \sum_{N=1}^{\infty} N \cdot x^{N-1}$$

$$= \sum_{N=0}^{\infty} (N+1) \cdot x^N$$

Bonus Problem A:

Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)}{2^n}$ $\longrightarrow$ plug in $x = \frac{1}{2}$ in the last example, so that

$$g\left(\tfrac{1}{2}\right) = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

Extra practice problem (added after the lecture today):

Find a power series for $\tan^{-1}(x)$ when $|x|<1$.

Solution:

$$\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2}, \text{ where for } |t|<1,$$

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt$$

$$= \sum_{N=0}^{\infty} (-1)^N \frac{t^{2N+1}}{2N+1} \Bigg|_0^x$$

$$= \sum_{N=0}^{\infty} (-1)^N \frac{x^{2N+1}}{2N+1}$$